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Weak and strong convergence theorems for a finite family of generalized asymptotically quasi-nonexpansive mappings

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ABSTRACT

In this paper, we introduce a new iterative scheme for finding a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. We establish weak and strong convergence theorems. Our main results improve and extend the corresponding ones obtained in Schu (1991) [J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mapping, J. Math. Anal. Appl. 159 (1991) 407–413] and many others.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be:

(i) *nonexpansive if* $\|Tx - Ty\| \leq \|x - y\|$ *for all* $x, y \in C$;

(ii) *quasi-nonexpansive if* $\|Tx - p\| \leq \|x - p\|$ *for all* $x \in C$ *and* $p \in F$;

(iii) *asymptotically nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$;

(iv) *asymptotically quasi-nonexpansive* if there exists a sequence $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(v) *generalized asymptotically nonexpansive* [1] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + c_n,$$

for all $x, y \in C$ and $n \geq 1$;

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(vi) *generalized asymptotically quasi-nonexpansive* [1] if there exist nonnegative real sequences $\{k_n\}$ and $\{c_n\}$ with $k_n \geq 1$, $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} c_n = 0$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| + c_n,$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$;

(vii) *uniformly L-Lipschitzian if there exists a constant* $L > 0$ *such that*

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ *and* $n \geq 1$.

From the above definitions, it is clear that:

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(iii) an asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(iv) an asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;
(v) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

However, the converse of each of above statements may be not true. A generalized asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping; see [1].

The map $T : C \to C$ is said to be:

(i) *demiclosed at 0 if for each sequence* $\{x_n\}$ *in* C *converging weakly to* x *and* $\{Tx_n\}$ *converging strongly to 0, we have* $Tx = 0$;
(ii) *semi-compact if for a sequence* $\{x_n\}$ *in* C *with* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *there exists a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *such that* $x_{n_j} \to p \in C$;
(iii) *completely continuous if for every bounded sequence* $\{x_n\} \subset C$, *there is a subsequence* $\{x_{n_k}\}$ *such that* $\{Tx_{n_k}\}$ *is convergent*.

A Banach space X is said to satisfy *Opial's property* (see [2]) if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

In 1991, Schu [3] introduced an iterative process:

$$x_0 \in C, \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1,$$

where $T : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in [0, 1] satisfying the condition $\delta \leq \alpha_n \leq 1 - \delta$ for all $n \in N$ and for some $\delta > 0$. He concluded that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Since 1972, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive type mappings in a Hilbert space or a Banach space have been studied by many authors (see, for example, [3–11]).

The problem of approximating common fixed points of finitely many mappings plays an important role in applied mathematics, especially in the theory of equations and the minimization problem; see [11–13] for example.

In 2008, Khan et al. [5] introduced and studied the following iterative process for a finite family of mappings $\{T_i : i = 1, 2, \ldots, k\}$ of C into itself: $x_0 \in C$,

$$\begin{aligned}
x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn} T_k^n y_{(k-1)n},\\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n} T_{k-1}^n y_{(k-2)n},\\
y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n} T_{k-2}^n y_{(k-3)n},\\
&\vdots\\
y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n} T_2^n y_{1n},\\
y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n} T_1^n y_{0n},
\end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

In 2010, Khan and Ahmed [6] introduced and studied the iterative process (1.1) in convex metric spaces and established its strong convergence to a unique common fixed point of a finite family of asymptotically quasi-nonexpansive mappings.

In 2010, Xiao et al. [7] modified the iterative process (1.1) to a $(k + 1)$-step iterative scheme with error terms involving $(k+1)$ asymptotically quasi-nonexpansive mappings and gave some sufficient and necessary conditions for approximating common fixed points of the mappings.

In this paper, we introduce a new iteration process for a finite family $\{T_i : i = 1, 2, \ldots, m\}$ of generalized asymptotically quasi-nonexpansive mappings of C into itself as follows:

$$x_0 \in C, \qquad x_{n+1} = S_n x_n, \quad \forall n \geq 1, \tag{1.2}$$

where $S_n = \alpha_{0n}I + \alpha_{1n}T_1^n + \alpha_{2n}T_2^n + \cdots + \alpha_{mn}T_m^n$ with $\alpha_{in} \in [0, 1]$, $\forall i = 0, 1, \ldots, m$, and $\sum_{i=0}^{m} \alpha_{in} = 1$. Weak and strong convergence of the sequence $\{x_n\}$ to a common fixed point of T_i $(i = 1, 2, \ldots, m)$ under some appropriate conditions, in a uniformly convex Banach space, are given.

Clearly, the iteration processes (1.1) and (1.2) generalize the modified Mann iteration from one mapping to the finite family of mappings $\{T_i : i = 1, 2, \ldots, m\}$ and so our work extends the results of Schu [3].

2. Main results

In this section, we establish some weak and strong convergence results for the iterative scheme (1.2) under some suitable conditions. To do this, we need the following lemmas.

Lemma 2.1 ([9]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers such that $b_n \geq 1$ and*

$$a_{n+1} \leq b_n a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty}(b_n - 1) < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2 ([10]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.3 ([14]). *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda)g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0, 1]$ where $\omega_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

By using Lemma 2.3, we can prove the following lemmas by induction.

Lemma 2.4. *Let X be a uniformly convex Banach space and $B_r(0) = \{ x \in X : \|x\| \leq r\}$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{m} \alpha_i x_i\right\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|),$$

for all $m \in N$, $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$, $i = 1, 2, \ldots, m$, with $\sum_{i=1}^{m} \alpha_i = 1$.

By interchanging the role of vectors x_i for all $i = 1, 2, \ldots, m$ in Lemma 2.4 and summing the inequalities together, we obtain the following.

Lemma 2.5. *Let X be a uniformly convex Banach space. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for each $m \in N$ and $j \in \{1, 2, \ldots, m\}$,*

$$\left\|\sum_{i=1}^{m} \alpha_i x_i\right\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \frac{\alpha_j}{m-1}\left(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|)\right),$$

for all $x_i \in B_r(0)$ and $\alpha_i \in [0, 1]$ for all $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

Proof. Let $j \in \{1, 2, \ldots, m\}$ be fixed. By Lemma 2.4, there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \cdots + \alpha_mx_m\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_j\alpha_1 g(\|x_j - x_1\|)$$

$$\|\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \cdots + \alpha_mx_m\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_j\alpha_2 g(\|x_j - x_2\|)$$

$$\vdots$$

$$\|\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \cdots + \alpha_mx_m\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_j\alpha_{j-1} g(\|x_j - x_{j-1}\|)$$

$$\|\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \cdots + \alpha_mx_m\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_j\alpha_{j+1} g(\|x_j - x_{j+1}\|)$$

$$\vdots$$

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_m x_m\|^2 \le \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_m g(\|x_j - x_m\|).$$

By summing the above inequalities, we obtain

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\|^2 \le \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1} \left(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|) \right). \quad \square$$

Lemma 2.6. *Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, m\}$ a family of generalized asymptotically quasi-nonexpansive self-mappings of C. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$ where $k_n = \max_{1\le i\le m} k_{in}$ and $c_n = \max_{1\le i\le m} c_{in}$. Let $\{x_n\}$ be the sequence defined by (1.2). Then:*

(i) *there exist two sequences $\{\delta_n\}$, $\{\epsilon_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\|x_{n+1} - p\| \le (1 + \delta_n)\|x_n - p\| + \epsilon_n$ for all $p \in F$ and $n \ge 1$;*

(ii) *there exist constants $M, S > 0$ such that $\|x_{n+k} - p\| \le M\|x_n - p\| + S$ for all $p \in F$ and $n, k \in N$.*

Proof. (i) Let $p \in F$, $k_n = \max_{1\le i\le m} k_{in}$ and $c_n = \max_{1\le i\le m} c_{in}$.

Now, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_n x_n - p\| \\
&= \|\alpha_{0n} x_n + \alpha_{1n} T_1^n x_n + \alpha_{2n} T_2^n x_n + \cdots + \alpha_{mn} T_m^n x_n - p\| \\
&\le \alpha_{0n}\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| + \alpha_{2n}\|T_2^n x_n - p\| + \cdots + \alpha_{mn}\|T_m^n x_n - p\| \\
&\le \alpha_{0n}\|x_n - p\| + \alpha_{1n}\left(k_{1n}\|x_n - p\| + c_{1n}\right) + \alpha_{2n}\left(k_{2n}\|x_n - p\| + c_{2n}\right) + \cdots + \alpha_{mn}\left(k_{mn}\|x_n - p\| + c_{mn}\right) \\
&\le \left(\alpha_{0n} + \alpha_{1n}k_{1n} + \alpha_{2n}k_{2n} + \cdots + \alpha_{mn}k_{mn}\right)\|x_n - p\| + c_{1n} + c_{2n} + \cdots + c_{mn} \\
&= \left(1 + \alpha_{1n}(k_{1n} - 1) + \alpha_{2n}(k_{2n} - 1) + \cdots + \alpha_{mn}(k_{mn} - 1)\right)\|x_n - p\| + c_{1n} + c_{2n} + \cdots + c_{mn} \\
&\le \left(1 + (\alpha_{1n} + \alpha_{2n} + \cdots + \alpha_{mn})(k_n - 1)\right)\|x_n - p\| + mc_n \\
&\le k_n\|x_n - p\| + mc_n.
\end{aligned} \tag{2.1}$$

Put $\delta_n = k_n - 1$ and $\epsilon_n = mc_n$. By our assumptions, it follows that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Hence, (2.1) reduces to $\|x_{n+1} - p\| \le (1 + \delta_n)\|x_n - p\| + \epsilon_n$. This completes the proof of (i).

(ii) If $t \ge 0$ then $1 + t \le e^t$. Thus, from part (i), for each $n, k \in N$, we get

$$\begin{aligned}
\|x_{n+k} - p\| &\le (1 + \delta_{n+k-1})\|x_{n+k-1} - p\| + \epsilon_{n+k-1} \\
&\le \exp\{\delta_{n+k-1}\}\|x_{n+k-1} - p\| + \epsilon_{n+k-1} \\
&\le \exp\{\delta_{n+k-1}\}\left((1 + \delta_{n+k-2})\|x_{n+k-2} - p\| + \epsilon_{n+k-2}\right) + \epsilon_{n+k-1} \\
&\le \exp\{\delta_{n+k-1}\}\exp\{\delta_{n+k-2}\}\|x_{n+k-2} - p\| + \exp\{\delta_{n+k-1}\}\epsilon_{n+k-2} + \epsilon_{n+k-1} \\
&\ \ \vdots \\
&\le \exp\left\{\sum_{i=0}^{k-1} \delta_{n+i}\right\}\|x_n - p\| + \exp\left\{\sum_{i=1}^{k-1} \delta_{n+i}\right\}\epsilon_n + \cdots \\
&\quad + \exp\{\delta_{n+k-1}\}\exp\{\delta_{n+k-2}\}\epsilon_{n+k-3} + \exp\{\delta_{n+k-1}\}\epsilon_{n+k-2} + \epsilon_{n+k-1} \\
&\le \exp\left\{\sum_{i=0}^{k-1} \delta_{n+i}\right\}\|x_n - p\| + \exp\left\{\sum_{i=1}^{k-1} \delta_{n+i}\right\}\sum_{i=0}^{k-1} \epsilon_{n+i}.
\end{aligned}$$

Setting $M = \exp\left\{\sum_{i=1}^{\infty} \delta_i\right\}$ and $S = \exp\left\{\sum_{i=1}^{\infty} \delta_i\right\}\sum_{i=1}^{\infty} \epsilon_i$, then $\|x_{n+k} - p\| \le M\|x_n - p\| + S$. Thus (ii) is satisfied. $\square$

Lemma 2.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{T_i : i = 1, 2, \ldots, m\}$ a family of uniformly L_i-Lipschitzian and generalized asymptotically quasi-nonexpansive self-mappings of C. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the following conditions hold:*

(C1) $\liminf_{n\to\infty} \alpha_{0n}\alpha_{in} > 0$, $\forall i = 1, 2, \ldots, m$.

(C2) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$ *where* $k_n = \max_{1\le i\le m} k_{in}$ *and* $c_n = \max_{1\le i\le m} c_{in}$. *Define the sequence $\{x_n\}$ as in (1.2). Then we have:*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists*, $\forall p \in F$;

(ii) $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$, *for each* $i = 1, 2, \ldots, m$;

(iii) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, *for each* $i = 1, 2, \ldots, m$.

Proof. Let $p \in F$, $k_n = \max_{1\le i\le m} k_{in}$ and $c_n = \max_{1\le i\le m} c_{in}$.

(i) By Lemmas 2.1 and 2.6(i), we obtain that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(ii) From (i), we have that $\{x_n\}$ is bounded. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\|T_i^n x_n - p\| &\le k_{in}\|x_n - p\| + c_{in}\\ &\le k_n\|x_n - p\| + c_n.\end{aligned}$$

Hence $\{T_i^n x_n - p\}$ is bounded $\forall i = 1, 2, \ldots, m$. Put $r = \sup\{\|T_i^n x_n - p\| : 1 \le i \le m, n \in N\} + \sup\{\|x_n - p\| : n \in N\}$. Let $i \in \{1, 2, \ldots, m\}$. By Lemma 2.5, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\left\|\sum_{j=0}^{i}\alpha_j y_j\right\|^2 \le \sum_{j=0}^{i}\alpha_j\|y_j\|^2 - \frac{\alpha_0}{i}\left(\sum_{j=1}^{i}\alpha_j g(\|y_0 - y_j\|)\right), \tag{2.2}$$

for all $y_j \in B_r(0)$ and $\alpha_j \in [0, 1]$, $\forall j = 0, 1, \ldots, i$ with $\sum_{j=0}^{i}\alpha_j = 1$. By (2.2), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_{0n}(x_n - p) + \alpha_{1n}(T_1^n x_n - p) + \cdots + \alpha_{mn}(T_m^n x_n - p)\|^2\\
&\le \alpha_{0n}\|x_n - p\|^2 + \alpha_{1n}\left(k_{1n}\|x_n - p\| + c_{1n}\right)^2 + \cdots + \alpha_{mn}\left(k_{mn}\|x_n - p\| + c_{mn}\right)^2\\
&\quad - \frac{\alpha_0}{m}\left(\sum_{j=1}^{m}\alpha_j g(\|x_n - T_j^n x_n\|)\right)\\
&\le \alpha_{0n}\|x_n - p\|^2 + \sum_{j=1}^{m}\alpha_{jn}\left(1 + (k_{jn} - 1)\right)^2\|x_n - p\|^2 + 2\sum_{j=1}^{m}\alpha_{jn}k_{jn}c_{jn}\|x_n - p\|\\
&\quad + \sum_{j=1}^{m}\alpha_{jn}c_{jn}^2 - \frac{\alpha_0}{m}\left(\sum_{j=1}^{m}\alpha_j g(\|x_n - T_j^n x_n\|)\right)\\
&\le \|x_n - p\|^2 + 2\sum_{j=1}^{m}\alpha_{jn}(k_{jn} - 1)\|x_n - p\|^2 + \sum_{j=1}^{m}\alpha_{jn}(k_{jn} - 1)^2\|x_n - p\|^2 + 2\sum_{j=1}^{m}\alpha_{jn}k_{jn}c_{jn}\|x_n - p\|\\
&\quad + \sum_{j=1}^{m}\alpha_{jn}c_{jn}^2 - \frac{\alpha_0}{m}\left(\sum_{j=1}^{m}\alpha_j g(\|x_n - T_j^n x_n\|)\right).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\alpha_{0n}}{m}\sum_{j=1}^{m}\alpha_{jn}g(\|x_n - T_j^n x_n\|) &\le \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) + \sum_{j=1}^{m}\alpha_{jn}(k_{jn} - 1)^2\|x_n - p\|^2\\
&\quad + 2\sum_{j=1}^{m}\alpha_{jn}(k_{jn} - 1)\|x_n - p\|^2 + 2\sum_{j=1}^{m}\alpha_{jn}k_{jn}c_{jn}\|x_n - p\| + \sum_{j=1}^{m}\alpha_{jn}c_{jn}^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, $\lim_{n\to\infty} k_{in} = 1$, $\lim_{n\to\infty} c_{in} = 0$ and $\liminf_{n\to\infty}\alpha_{0n}\alpha_{in} > 0$ for each $i = 1, 2, \ldots, m$, it follows that $\lim_{n\to\infty} g(\|x_n - T_i^n x_n\|) = 0$. Since g is continuous strictly increasing with $g(0) = 0$, we can conclude that

$$\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0, \quad \forall i = 1, 2, \ldots, m. \tag{2.3}$$

(iii) For each $n \in N$, we have

$$\begin{aligned}\|x_{n+1} - T_1^n x_n\| &= \|\alpha_{0n}x_n + \alpha_{1n}T_1^n x_n + \alpha_{2n}T_2^n x_n + \cdots + \alpha_{mn}T_m^n x_n - T_1^n x_n\|\\
&\le \alpha_{0n}\|x_n - T_1^n x_n\| + \alpha_{2n}\|T_2^n x_n - T_1^n x_n\| + \cdots + \alpha_{mn}\|T_m^n x_n - T_1^n x_n\|\\
&\le \alpha_{0n}\|x_n - T_1^n x_n\| + \alpha_{2n}\|T_2^n x_n - x_n\| + \alpha_{2n}\|x_n - T_1^n x_n\| + \cdots + \alpha_{mn}\|T_m^n x_n - x_n\| + \alpha_{mn}\|x_n - T_1^n x_n\|\\
&= (1 - \alpha_{1n})\|x_n - T_1^n x_n\| + \alpha_{2n}\|T_2^n x_n - x_n\| + \cdots + \alpha_{mn}\|T_m^n x_n - x_n\|.\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have $\lim_{n\to\infty}\|x_{n+1} - T_1^n x_n\| = 0$. This implies that

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - T_1^n x_n\| + \|T_1^n x_n - x_n\| \to 0. \tag{2.5}$$

For each $i = 1, 2, \ldots, m$, we have

$$\|T_i x_{n+1} - x_{n+1}\| \le \|T_i x_{n+1} - T_i^{n+1}x_{n+1}\| + \|T_i^{n+1}x_{n+1} - x_{n+1}\|$$

$$\begin{aligned}
&\le L_i\|x_{n+1} - T_i^n x_{n+1}\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\
&\le L_i(\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|) + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\
&\le L_i(\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + L_i\|x_n - x_{n+1}\|) + \|T_i^{n+1} x_{n+1} - x_{n+1}\|.
\end{aligned}$$

From (2.3) and (2.5), we obtain that $\|T_i x_{n+1} - x_{n+1}\| \to 0,\ \forall i = 1, 2, \dots, m$. This implies that $\|T_i x_n - x_n\| \to 0,\ \forall i = 1, 2, \dots, m$. Thus (iii) is satisfied. □

Theorem 2.8. *Under the hypotheses of Lemma 2.7, assume that one of T_i is completely continuous. Then the iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, m\}$.*

Proof. Suppose that T_{i_0} is completely continuous for some $i_0 \in \{1, 2, \dots, m\}$. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $T_{i_0} x_{n_k} \to p \in C$. By Lemma 2.7(iii), we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \forall i = 1, 2, \dots, m$. It follows that

$$\|x_{n_k} - p\| \le \|x_{n_k} - T_{i_0} x_{n_k}\| + \|T_{i_0} x_{n_k} - p\| \to 0.$$

Thus $x_{n_k} \to p$. By the continuity of T_i, we have

$$\|p - T_i p\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0, \quad \forall i = 1, 2, \dots, m.$$

Hence $p \in F$. By Lemma 2.7(i), we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. □

Theorem 2.9. *Under the hypotheses of Lemma 2.7, assume that one of the T_i is semi-compact. Then the iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, m\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \dots, m\}$. By Lemma 2.7(iii), we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $\forall i = 1, 2, \dots, m$. Since $\{x_n\}$ is bounded and T_{i_0} is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to p \in C$. By the continuity of T_i, we have

$$\|p - T_i p\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0, \quad \forall i = 1, 2, \dots, m.$$

Thus $p \in F$. By Lemma 2.7(i), we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\lim_{n\to\infty} \|x_n - p\| = 0$. □

Theorem 2.10. *Under the hypotheses of Lemma 2.7, assume that $(I - T_i)$ is demiclosed at 0, for each $i = 1, 2, \dots, m$. Then $\{x_n\}$ in (1.2) converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, m\}$.*

Proof. Let $p \in F$. By Lemma 2.7(i), $\lim_{n\to\infty} \|x_n - p\|$ exists, and hence $\{x_n\}$ is bounded. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $q_1 \in C$. By Lemma 2.7(ii), $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$. Since $(I - T_i)$ is demiclosed at 0 for each $i = 1, 2, \dots, m$, we obtain that $T_i q_1 = q_1$. That is, $q_1 \in F$. Next, we show that $\{x_n\}$ converges weakly to q_1. Take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q_2 \in C$. Again, as above, we can conclude that $q_2 \in F$. By Lemma 2.2, we obtain that $q_1 = q_2$. This shows that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, m\}$. □

Remark 2.11. The main results of [6,7] gave weak and strong convergence theorems for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings by using the iterative scheme (1.1) and its modification which is a finite step iteration, while the main results of this paper give weak and strong convergence theorems for finding a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings by using the iterative scheme (1.2) which is simpler than that of (1.1). Moreover, the class of mappings studied in this paper is more general than the class of mappings studied in [6,7].

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